Math 53: Multivariable Calculus **Sections 102, 103**

Handout for 2020-03-02

Problem 1. Consider the function $f(x, y) = \sqrt{1 + xy}$. Let r, θ denote polar coordinates for the plane. Use the chain rule to compute $\partial f/\partial r$ and $\partial f/\partial\theta$ at $(x, y) = (6, 8)$, where $r = 10$. (You could also do this without the chain rule.)

Problem 2. Let $f(x, y)$ and $g(u, v)$ be two functions, related by

$$g(u, v) = f(e^u + \sin v, e^u + \cos v).$$

Use the following values to calculate $g_u(0, 0)$ and $g_v(0, 0)$ (not all of the below values may be relevant!).

$f(0,0) = 3$	$g(0,0) = 6$	$f_x(0,0) = 4$	$f_y(0,0) = 8$
$f(1,2) = 6$	$g(1,2) = 3$	$f_x(1,2) = 2$	$f_y(1,2) = 5$

Problem 3.

(a) Show that the equation

$$x^7 - ax^6 + bx - 2 = 0$$

has a solution x as a function of a, b for (a, b) near $(1, 2)$ and x near 1.

(b) If we instead consider the equation

$$x^7 - 1.03x^6 + 2.06x - 2 = 0,$$

which is very similar to the original equation, a "zeroth order" approximation to a root would be $x = 1$ since that was a root of the original equation. If you plug that in to the left hand side, you get 0.03. We can do better: use implicit differentiation to find a "first order" (linear) approximation of a solution to the above equation.

Problem 4. Fix a nonnegative integer a and consider the function

$$f(x, y) = \begin{cases} \dfrac{(x + y)^a}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

For what choices of a is the function f continuous?

Problem 5. For $F(x, y) = 0$, we know that if $F_y \neq 0$ then this defines y implicitly as a function of x and $\mathrm{d}y/\mathrm{d}x = -F_x/F_y$. Find a formula for $\mathrm{d}^2y/\mathrm{d}x^2$ in terms of the partial derivatives of F.

Problem 1. Use the chain rule to find that

$$\left.\frac{\partial f}{\partial r}\right|_{x=6,y=8,r=10} = \boxed{\frac{24}{35}}$$
$$\left.\frac{\partial f}{\partial \theta}\right|_{x=6,y=8,r=10} = \boxed{-2}.$$

Problem 2. Let $x = e^u + \sin v$ and $y = e^u + \cos v$. Note that when $(u, v) = (0, 0)$, $(x, y) = (1, 2)$. So:

$$g_u(0,0) = f_x(1,2)e^0 + f_y(1,2)e^0 = \boxed{7}$$
$$g_v(0,0) = f_x(1,2)\cos(0) + f_y(1,2)(-\sin 0) = \boxed{2}.$$

Problem 3. (a) Let $F(a, b, x) = x^7 - ax^6 + bx - 2$. First one checks that $F(1, 2, 1) = 0$. Then, by the Implicit Function Theorem, it suffices to show that $F_x(1, 2, 1) \neq 0$:

$$F_x(1,2,1) = 7 - 6 + 2 = 3.$$

Thus the equation describes x as a differentiable function of a, b near our point.

(b) This is just a matter of computing $\partial x/\partial a$ and $\partial x/\partial b$, which you will find to be 1/3 and −1/3 respectively. So:

$$x(1.03, 2.06) \approx 1 + \frac{1}{3}(0.03) + \frac{-1}{3}(0.06) = \boxed{0.99}.$$

Problem 4. In other words, when is

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^a}{x^2+y^2} = 0?$$

It turns out that this limit does not exist when $a = 0, 1, 2$ (testing along lines is sufficient). The limit does exist and is equal to zero when $a \geq 3$; perhaps this is most easily seen with polar coordinates and the Squeeze Theorem.

Problem 5. The answer to this problem can be found in Exercise 59 on page 946 of Stewart. The idea is to differentiate $-F_x/F_y$ with respect to x, noting that the expression involves both x and y and thus the chain rule is again necessary. Some expressions $\mathrm{d}y/\mathrm{d}x$ will show up when you do this—but you already have a formula for that!